UNIVERSAL POSITIVE DEFINITE KERNELS

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Positive definite kernels appear as important tools in many branches of mathematics such as linear algebra, functional analysis, approximation theory, just to mention a few (see [2]). In particular, the modern learning theory was built upon some key properties of the positive definite kernels (see [3]).

Perhaps, the most important property a positive definite kernel $K : E \times E \to \mathbb{C}$, where E is a topological space, has is this (see [1]): there exists a Hilbert space $(\mathcal{W}, \langle \cdot, \cdot \rangle)$ and a mapping $\phi : E \to \mathcal{W}$ such that

$$K(x, y) = \langle \phi(x), \phi(y) \rangle\,, \quad x, y \in E. \tag{1}$$

The function ϕ is usually called a *feature map* of K while $\mathcal{W}$ is called the *feature space*. It is well known that $\mathcal{W}$ and ϕ are not unique.

A continuous kernel K as above has *the universal property for approximation* (that is, K is universal) if for every $\varepsilon > 0$ and every compact subset X of E the following property holds: if f is a continuous function on X, there exists a function g in the closure $G_K(X)$ of $[\{K(\cdot, y) : y \in X\}]$ in $C(X)$, the space of all complex valued continuous functions on X, endowed with its uniform norm $\| \cdot \|_\infty$, so that $\|f - g\|_\infty < \varepsilon$. Since $G_K(X)$ is closed in $C(X)$, the requirement above corresponds to $G_K(X) = C(X)$.

Let K be a continuous kernel as above and consider an orthonormal basis $\mathcal{B}$ of $\mathcal{W}$ (the existence of that is guaranteed in [4, p.144]). The feature map ϕ is *universal* if for every $\varepsilon > 0$, every compact subset X of E and every f in $C(X)$, there exists a function g in the closure $\phi(\mathcal{B})$ of $[\{\langle \phi(\cdot), v \rangle : v \in \mathcal{B}\}]$ in $C(X)$ such that $\|f - g\|_\infty < \varepsilon$. Here, the closeness of $\phi(\mathcal{B})$ in $C(X)$, reveals that the requirement above corresponds to $\phi(\mathcal{B}) = C(X)$.

The intended goal in this talk is to prove the following relationship between the two concepts introduced above: a continuous and positive definite kernel $K : E \times E \to \mathbb{C}$ is universal if and only if the corresponding feature map ϕ is so.

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References

[1] Micchelli, Charles A.; Xu, Yuesheng; Zhang, Haizhang *Universal kernels.* J. Mach. Learn. Res. 7, 2006, 2651 - 2667.

[2] Cheney, W.; Light, W., *A course in approximation theory.* Reprint of the 2000 original. Graduate Studies in Mathematics. American Mathematical Society, Providence, RI, 2009.

[3] Cucker, F.; Zhou, Ding-Xuan, *Learning theory: an approximation theory viewpoint.* With a foreword by Stephen Smale. Cambridge Monographs on Applied and Computational Mathematics. Cambridge University Press, Cambridge, 2007.

[4] Bachman, George; Narici, Lawrence, *Functional analysis.* Academic Press, New York-London 1966.

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