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# THE NILPOTENT CENTER-FOCUS PROBLEM

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One of the main and oldest problems in the qualitative theory of differential systems in $\mathbb{R}^2$ is the distinction between a center and a focus, called the *the center-focus problem*.

Let $p \in \mathbb{R}^2$ be a singular point of a planar analytic differential system, and assume that $p$ is a center. Without loss of generality, we can assume that $p$ is the origin. After a linear change of variable and a rescaling of the time variable (if necessary), the system can be written in one of the following three forms:

$$\dot{x} = -y + F_1(x, y), \qquad \dot{y} = x + F_2(x, y); \tag{0.1}$$

$$\dot{x} = y + F_1(x, y), \qquad \dot{y} = F_2(x, y); \tag{0.2}$$

$$\dot{x} = F_1(x, y), \qquad \dot{y} = F_2(x, y); \tag{0.3}$$

where $F_1$ and $F_2$ are real analytic functions without constant and linear terms in a neighborhood of the origin. We call a center of an analytic differential system in $\mathbb{R}^2$ *linear type, nilpotent or degenerate* if after an affine change of variables and rescaling of the time it can be written as (0.1), (0.2) or (0.3), respectively.

[1] and Lyapunov [2], see also Moussu [3] characterized linear type centers in terms of the existence of an analytic first integral. Lyapunov introduced the concept of the function now known as *Lyapunov functions*, as well as the *Lyapunov constants*. These last constants give the stability of the singular points, in particular, the center case is characterized by the vanishing of all the Lyapunov constants. Therefore, theoretically, the center-focus problem for linear type centers is solved. Nevertheless, despite of the big efforts done in the last years with the appearance of new algorithms and the remarkable increase in computational power, there are still big difficulties in the complete solution of the problem when a particular family of differential equations is given. Even in the case of polynomial systems of a given degree, for which the Hilbert Basis Theorem asserts that the number of needed Lyapunov constants is finite, it is neither known which is this number.

Until now there is no algorithm comparable to the Poincaré-Lyapunov method for determining the center conditions in the case of nilpotent and degenerate singular points. In this presentation we present the method developed in [4] and [5], in order to characterize the center-focus problem for nilpotent critical points. We apply the method in a well known family of planar analytic vector fields and discuss the computational and the sufficiency obstructions that arise from this approach.

## References

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