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REGULARITY AT INFINITY OF REAL MAPPINGS

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Let $f\colon \mathbb{K}^n \to \mathbb{K}^p$, for $n > p > 0$ and $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$, be a non-constant polynomial mapping. Let y be a point in $\mathbb{K}^p$. We say that f is *topologically trivial at y* if there exist a neighborhood U of y and a homeomorphism $h\colon f^{-1}(y) \times U \to f^{-1}(U)$ such that $f \circ h = pr_2$, where $pr_2\colon f^{-1}(y) \times U \to U$ is the second projection. It is well-known that f is topologically trivial over the complement of the *bifurcation set* $B(f)$, also called *the set of atypical values.* The atypical values may come from the critical values but also from the asymptotic behavior of the fibers. One can easily see this in the example $f(x, y) = x + x^2y$, where the value $0 \in \mathbb{K}$ is not critical but f is not topologically trivial at 0.

A complete characterization of $B(f)$ is available only in the case $n = 2$ and $p = 1$, see for instance [4] for $\mathbb{K} = \mathbb{C}$ and [6] for $\mathbb{K} = \mathbb{R}$. One has therefore imagined various ways to approximate $B(f)$, essentially through the use of *regularity conditions at infinity.*

In [3], Kurdyka, Orro and Simon considered C^2 semi-algebraic maps $f : \mathbb{K}^n \to \mathbb{K}^p$ and a metric type regularity condition, which we shall call here *KOS-regularity.* They define a set of asymptotically critical values $K_\infty(f)$ and prove that $K_\infty(f)$ is a semi-algebraic set of dimension $< p$. Based on the results of [5], they show $A_{KOS} \supset B(f)$, where $A_{KOS} := f(\mathrm{Sing} f) \cup K_\infty(f)$.

Gaffney [2], considers polynomial mappings $f\colon \mathbb{C}^n \to \mathbb{C}^p$. He defines the *generalized Malgrange condition* and proves that this yields a set A_G of non-regular values containing $B(f)$. He uses the theory of integral closure of modules to relate his condition to a non-characteristic condition like Parusiński [4].

In this talk, we consider two regularity conditions, the *t-regularity*, which is a geometric condition, and the *ρ_E-regularity*, a Milnor type condition. We show that the t-regularity is equivalent to the KOS-regularity and, based on the existence of *partial Thom stratifications at infinity*, we give a different proof that A_{KOS} has dimension $< p$.

Following Gaffney, we give an interpretation of t-regularity in terms of integral closure of modules. This allows one to prove that t-regularity is equivalent to the generalized Malgrange condition reformulated over $\mathbb{R}$.

We pursue by showing that t-regularity implies *ρ_E-regularity.* The ρ_E-regularity enables one to define the set of asymptotic non ρ_E-regular values $S(f) \subset \mathbb{R}^p$, and the set $A_{\rho_E} := f(\mathrm{Sing} f) \cup S(f)$.

Then, we show that the subset $S(f)$ is closed semi-algebraic and of dimension $< p$. Moreover, we show that f is topologically trivial on the complement $\mathbb{R}^p \setminus A_{\rho_E}$. This result is based on a fibration theorem "at infinity", i.e. holding in the complement of a sufficiently large ball.

This work is based on [1].

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