I Workshop de Teses e Dissertações em Matemática
ICMC - Instituto de Ciências Matemáticas e de Computação
USP - Universidade de São Paulo
December 1-2, 2011

geometric Weyl quantization

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1 Introduction

Following on the quantization by groupoids program as a way to obtain an integral product which would deform the multiplication of the Poisson algebra of functions on a symplectic manifold M as in Weinstein [5]. Me searches a product of the general form

$$(fg)(z) = \int_{M\times M} f(x)g(y)K(x,y,z)dxdy,$$

with a kernel $K_\hbar$, depending on the deformation parameter $\hbar$, of the kind $K_\hbar(x,y,z) = \hbar^{dimM}.exp(iS(x,y,z)/\hbar)$, eventually multiplied by an "amplitude" $A(x,y,z)$, where the function $S(x,y,z)$ could be the symplectic area of a surface whose boundary is the geodesic triangle for which the points x, y and z are the midpoints of its sides, generalizing what is know for $\mathbb{R}^{2n}$.

In this present work, we will derive such a formula(Moyal-Weyl integral product) for $\mathbb{R}^2$ with the euclidian connection and for $\mathbb{R}^2$ how a (non-metric) symplectic symmetric space(see [1] for details), by means of geometric quantization of the symplectic groupoid $M\times M$ and its prequantization as described in [6, 3].

2 Mathematical Results

Let (M,ω) be a symplectic manifold and let $\hbar \in \mathbb{R}^+$ be a parameter. Let (Y,θ) be a prequantization of $(M,\omega/\hbar)$, meaning that $\pi : Y \to M$ is a principal S^1-bundle equipped with a connection form θ whose curvature is $\omega/\hbar$. We let $L \to M$ be the associated complex line bundle over M with connection ∇ and compatible hermitian structure. It follows that we can identify Y with the subset of L of points of length 1. Now, consider a prequantization of the pair groupoid $M\times M$ the symplectic structure $(\omega,-\omega)$. We let $[L] \to M\times M$ be the associated complex line bundle with connection and compatible hermitian structure (see [2] for details).

Theorem 2.1. *Let the symplectic groupoid* $G = \mathbb{R}^2 \times \overline{\mathbb{R}}^2$ *of coordinates* $(x_1, y_1; x_2, y_2)$*, with which its symplectic form may be written as*

$$\Omega = dx_1 \wedge dy_1 - dx_2 \wedge dy_2.$$

If P *is a central (real)polarization and* F *is a trivial (real)polarization of the* $\mathbb{R}^2 \times \overline{\mathbb{R}}^2$ *such that* F *and* P *are nontransverse. Then for the two sections* $\alpha \in \Gamma_F([L]\otimes Q^F)$ *and* $\beta \in \Gamma_P([L]\otimes Q^P)$

$$<\alpha,\beta>_h = <f, Tg>_{L^2(\mathbb{R}^2)}, \quad where$$

- *In the case of the Euclidean plane, we have:* $Tg(x_1,x_2) = (2\pi\hbar)^{-1}\int_{\mathbb{R}} g(\frac{x_1-x_2}{2},\xi)exp(i\frac{\xi}{\hbar}(x_2-x_1))d\xi$
- *In the case of the Bieliavsky plane(see* [1] *) we have:*

$$Tg(x_1,x_2) = (2\pi\hbar)^{-1}\int_{\mathbb{R}} g(\frac{x_1+x_2}{2},\xi)exp(2i\frac{\xi}{\hbar}\sinh\frac{(x_2-x_1)}{2})\cosh^{\frac{1}{2}}(\frac{(x_2-x_1)}{2})d\xi$$

- $\alpha = f.s_0\otimes\sqrt{d\alpha_1}$ *and* $\beta = g.t_0\otimes\sqrt{d\beta_1}$*, such that* s_0, t_0 *are nonvanishing section of* L *and* $\sqrt{d\alpha_1}, \sqrt{d\beta_1}$ *are half-forms on* $\mathbb{R}^2\times\overline{\mathbb{R}}^2$ *associated with* P, F *respectively.*

- $< .,. >_h$ *is the pairing of two sections* $t \otimes \sqrt{\alpha_1} \in \Gamma_F([L] \otimes Q^F)$, $s \otimes \sqrt{\beta_1} \in \Gamma_P([L] \otimes Q^P)$ *given by* $< t \otimes \sqrt{\alpha_1}, s \otimes \sqrt{\beta_1} >_h = \int (t,s) < \sqrt{\alpha_1}, \sqrt{\beta_1} >_{PR}$, *where* $(.,.)$ *is the Hermitian metric on* $[L]$ *and* $< .,. >_{PR}$ *is the pairing of half-forms.*

Remark 2.1. *If the symplectic groupoid* $G = \mathbb{R}^2 \times \overline{\mathbb{R}}^2$ *has coordinates* $(x_1, y_1; x_2, y_2)$ *the polarization* F *above is generated for* $\{\partial y_1, \partial y_2\}$, *and the product groupoid is:* $(x_1, y_1; x_3, y_3) = (x_1, y_1; x_2, y_2).(x_2, y_2; x_3, y_3)$.

Let $\alpha = f.s_0 \otimes \sqrt{\alpha_1}$, $\beta = g.s_0 \otimes \sqrt{\beta_1}$ *in* $\Gamma_P([L] \otimes Q^P)$, *we define* $\Upsilon(\alpha) = Tf(x_1, x_2)s_0(x_1, y_1, x_2, y_2) \otimes \sqrt{dx_1 \wedge dx_2}$ *and* $\Upsilon(\beta) = Tg(x_2, x_3)s_0(x_2, y_2, x_3, y_3) \otimes \sqrt{dx_2 \wedge dx_3}$, *sections in the* $\Gamma_F([L] \otimes Q^F)$. *Analogously if* $\rho = h.t_0 \otimes \sqrt{\rho_1}$ *is a section in* $\Gamma_F([L] \otimes Q^F)$, *we define* $\Upsilon^{-1}(\rho) = T^{-1}h.s_0 \otimes \sqrt{\rho_1'}$ *a section in* $\Gamma_P([L] \otimes Q^P)$, *where* $\sqrt{\rho_1'}$ *is the generator of* Q^P. *Now consider a new section of* $[L]$ *by*

$$\begin{aligned} \Upsilon(\alpha) \circledcirc \Upsilon(\beta)(x_1, y_1, x_3, y_3) &= \int_{\mathbb{R}} Tf(x_1, x_2)Tg(x_2, x_3)t_0(x_1, y_1, x_2, y_2) \odot t_0(x_2, y_2, x_3, y_3)dx_2 \otimes \sqrt{dx_1 \wedge dx_3} \\ &= \int_{\mathbb{R}} Tf(x_1, x_2)Tg(x_2, x_3)dx_2 t_0(x_1, y_1, x_3, y_3) \otimes \sqrt{dx_1 \wedge dx_3}. \end{aligned}$$

Then $\Upsilon^{-1}(\Upsilon(\alpha) \circledcirc \Upsilon(\beta)) = (f \star_\hbar^0 g).s_0 \otimes \sqrt{\rho_1'}$, *where* $\star_\hbar^0$ *is the Moyal-Weyl product,i.e.*

$$(u \star_\hbar^0 v)(x) = \frac{1}{\hbar^{2n}} \int_{\mathbb{R}^{2n} \times \mathbb{R}^{2n}} u(y)v(z)e^{-\frac{2i}{\hbar}S^0(x,y,z)}$$

where

$$S^0(x, y, z) = \omega^0(x, y) + \omega^0(y, z) + \omega^0(z, x).$$

In the case of the Bieliavsky plane(see [1]) we have that $(u \star_\hbar^B v)(x_0)$ is:

$$\frac{1}{\hbar^2} \int_{\mathbb{R}^2 \times \mathbb{R}^2} \sqrt{\cosh(a_0 - a_1)\cosh(a_1 - a_2)\cosh(a_2 - a_0)} e^{\frac{i}{\hbar}[l_2 \sinh(a_0 - a_1) + l_1 \sinh(a_2 - a_0) + l_0 \sinh(a_1 - a_2)]} u(x_1)v(x_2)dx_1 dx_2$$

The basic information mentioned above about the polarization and prequantization is to be found in [4] while additional information on the central polarization, central polarized sections and the product section $\odot$ can be found in [2].

References

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