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GLOBAL SOLVABILITY FOR A CLASS OF INVOLUTIVE SYSTEMS

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1 Introduction

We consider a class of involutive systems of n vector fields on the torus $\mathbb{T}^{n+1} \simeq (\mathbb{R}/2\pi\mathbb{Z})^{n+1}$, given by

$$L_j = \frac{\partial}{\partial t_j} + (a_j(t) + ib_j(t_j))\frac{\partial}{\partial x}, \quad j = 1, \ldots, n, \tag{1.1}$$

where $a_j \in C^\infty(\mathbb{T}^n;\mathbb{R})$, $b_j \in C^\infty(\mathbb{T}^1;\mathbb{R})$ and $(t,x) = (t_1, \ldots, t_n, x)$ are the coordinates on the torus $\mathbb{T}^{n+1}$.

We assume that the system (1.1) is involutive or equivalently that the 1-form $c = a + ib \in \bigwedge^1 C^\infty(\mathbb{T}^n)$ is closed, where $a(t) = \sum_{j=1}^n a_j(t)dt_j$ and $b(t) = \sum_{j=1}^n b_j(t_j)dt_j$ are real 1-forms on $\mathbb{T}^n$. For further explanations about this concept see the book [4].

We obtain a complete characterization for the global solvability of this class in terms of Liouville forms and of the connectedness of all sublevel and superlevel sets of the primitive of pull-back of b in the minimal covering space.

2 Preliminaries and statement of the main results

We may write $c = c_0 + d_t\mathcal{C}$ where $\mathcal{C}$ is a complex valued smooth function of $t \in \mathbb{T}^n$ and $c_0 \in \bigwedge^1 \mathbb{C}^n \simeq \mathbb{C}^n$. If $f = (f_1, \ldots, f_n) \in C^\infty(\mathbb{T}^{n+1};\mathbb{C}^n)$ and if there exists $u \in \mathcal{D}'(\mathbb{T}^{n+1})$ such that $L_ju = f_j$, $j = 1, \ldots, n$, then $L_jf_k = L_kf_j$, $j,k = 1, \ldots, n$ and

$$\hat{f}(t,\xi)e^{i\xi(c_0\cdot t+\mathcal{C}(t))} \text{ is exact when } \xi \in \mathbb{Z} \text{ is such that } \xi c_0 \in \mathbb{Z}^n, \tag{2.2}$$

where $\hat{f}(t,\xi) \doteq \sum_{j=1}^n \hat{f}_j(t,\xi)dt_j$ and $\hat{f}_j(t,\xi)$ denotes the Fourier transform with respect to x. Therefore, we define the following set

$$\mathbb{E} = \big\{f = (f_1, \ldots, f_n) \in C^\infty(\mathbb{T}^{n+1};\mathbb{C}^n);\ L_jf_k = L_kf_j \text{ and (2.2) holds}\big\}.$$

Definition 2.1. *The system* (1.1) *is said to be globally solvable on* $\mathbb{T}^{n+1}$ *if for each* $f = (f_1, \ldots, f_n) \in \mathbb{E}$ *there exists* $u \in \mathcal{D}'(\mathbb{T}^{n+1})$ *satisfying* $L_ju = f_j$, $j = 1, \ldots, n$.

We identify the 1-form $c_0 \in \bigwedge^1 \mathbb{C}^n$ with the vector $(c_{10}, \ldots, c_{n0})$ in $\mathbb{C}^n$ consisting of the periods $c_{j0} = a_{j0} + ib_{j0}$ where $a_{j0} = \frac{1}{2\pi}\int_0^{2\pi} a_j(0, \ldots, \tau_j, \ldots, 0)d\tau_j$ and $b_{j0} = \frac{1}{2\pi}\int_0^{2\pi} b_j(\tau_j)d\tau_j$. Also, we use the notation $a_0 = (a_{10}, \ldots, a_{n0})$ and $b_0 = (b_{10}, \ldots, b_{n0})$.

Given $\alpha = (\alpha_1, \ldots, \alpha_N) \in \mathbb{R}^N$. If $\alpha \in \mathbb{Q}^N$, for each subset $J = \{j_1, \ldots, j_m\} \neq \emptyset$ of $\{1, \ldots, N\}$ we denote by q_J the smallest positive integer such that $q_J(\alpha_{j_1}, \ldots, \alpha_{j_m}) \in \mathbb{Z}^m$. When $J = \{1, \ldots, N\}$ we write $q_* \doteq q_J$; thus $q_J \leq q_*$ and q_J divides q_*.

If, otherwise, $\alpha \notin \mathbb{Q}^N$ we say that α is Liouville when there are a constant $C > 0$ and a sequence $\{(\kappa_l, \xi_l)\}$ in $\mathbb{Z}^N \times \mathbb{Z}$ ($\xi_l \geq 2$) such that

$$\max_{j=1,\ldots,N}\left|\alpha_j - \frac{\kappa_l^{(j)}}{\xi_l}\right| \leq \frac{C}{(\xi_l)^l},\ \forall l \in \mathbb{N}.$$

In [2] the authors define the *minimal* covering of $\mathbb{T}^n$ with respect to the 1-form b as the smallest covering space $\Pi : \mathcal{T} \to \mathbb{T}^n$ where the pull-back Π^*b is exact. The minimal covering of $\mathbb{T}^n$ is isomorphic to $\mathcal{T} = \mathbb{R}^r \times \mathbb{T}^{n-r}$

where $r = 0, 1 \ldots, n$ is the rank of the group of the periods of b. In the minimal covering the 1-form $\Pi^* b$ has a global primitive B and since each b_j depends only on the coordinate t_j the function $B : \mathcal{T} \to \mathbb{R}$ is of the form $B(t) = \sum_{j=1}^n B_j(t_j)$.

The main result of this work is the following theorem:

Theorem 2.1. *Let* $J \doteq \{j_1, \ldots, j_m\} = \{j \in \{1, \ldots, n\},\ b_j \equiv 0\}$ *and* B *a primitive of* $\Pi^* b$ *in the minimal covering* $\mathcal{T}$. *With the above notation, the system* (1.1) *is globally solvable if and only if one of the following two situations occurs:*

I) $J \neq \emptyset$ *and* $(a_{j_1 0}, \ldots, a_{j_m 0}) \notin \mathbb{Q}^m$ *is non-Liouville.*

II) The sublevels $\Omega_s = \{t \in \mathcal{T},\ B(t) = \sum_{j=1}^n B_j(t_j) < s\}$ *and superlevels* $\Omega^s = \{t \in \mathcal{T},\ B(t) = \sum_{j=1}^n B_j(t_j) > s\}$ *are connected for every* $s \in \mathbb{R}$ *and, additionally, one of the following conditions holds:*

1. $J = \emptyset$, *b is exact and* $a_0 \in \mathbb{Z}^n$;
2. $J \neq \emptyset$, *b is exact,* $a_0 \in \mathbb{Q}^n$ *and* $q_J = q_*$;
3. *b is not exact.*

In the proof of this theorem, we use the following important result:

Proposition 2.1. *If b is not exact and $\mathcal{T}$ denotes the minimal covering, then the sublevels* $\Omega_s = \{t \in \mathcal{T},\ B(t) = \sum_{j=1}^n B_j(t_j) < s\}$ *and superlevels* $\Omega^s = \{t \in \mathcal{T},\ B(t) = \sum_{j=1}^n B_j(t_j) > s\}$ *are connected for every* $s \in \mathbb{R}$ *if and only if there exists a function* $b_j \not\equiv 0$ *that does not change sign.*

Theorem 2.1 provides the following interesting example: The system

$$\begin{cases} L_1 = \frac{\partial}{\partial t_1} + \frac{1}{4}\frac{\partial}{\partial x} \\ L_2 = \frac{\partial}{\partial t_2} + (\frac{1}{2} + i\sin(t_2))\frac{\partial}{\partial x} \end{cases},$$

is globally solvable on $\mathbb{T}^3$ since $q_J = q_* = 4$, whereas

$$\begin{cases} L_1 = \frac{\partial}{\partial t_1} + \frac{1}{2}\frac{\partial}{\partial x} \\ L_2 = \frac{\partial}{\partial t_2} + (\frac{1}{4} + i\sin(t_2))\frac{\partial}{\partial x} \end{cases},$$

is not globally solvable since in this case $q_J = 2 < 4 = q_*$.

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