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Riemann Hypothesis and Physics

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The goal of this work is to discuss on the main ingredients that lead to the postulation of the Riemann Hypothesis. Our intention is to fill up the gap that makes that only a small portion of the mathematics community knows what makes the Riemann Hypothesis plausible, even though it is today one of the main open problems. We also present some interesting mathematical results related to the Riemann zeta function and the Riemann hypothesis, such as the Prime Number Theorem, as well as their relationships with physical models. The material covered here is far from being exhaustive. It is written in a way to allow undergraduate students in the field of sciences to understand it with little work. We begin with the complex variable Riemann zeta function

$$\zeta(s) = \sum_{n=1}^{+\infty} \frac{1}{n^s} \ , \quad s = \sigma + it,\ \sigma > 1,\ t \in \mathbb{R} \, , \tag{1}$$

which has many other representaions and extends to a meromorphic function of $s \in \mathbb{C}$, still denoted by $\zeta(s)$, with a simple pole at $s = 1$ and $\mathrm{Res}(\zeta, 1) = 1$. It obeys the functional equation

$$\zeta(s) = 2^s \pi^{s-1} \sin\left(\frac{\pi s}{2}\right) \Gamma(1-s)\zeta(1-s) \ , \quad s \in \mathbb{C}-\{1\} \, , \tag{2}$$

where Γ is the Euler gamma function. Due to the $\sin\left(\frac{\pi s}{2}\right)$ factor, Eq. (2) equation shows that $\zeta(s)$ has simple zeroes at $s = \ldots, -6, -4, -2$. These are the *trivial* zeroes of $\zeta(s)$. From Eq. (1), we can also prove that $\zeta(s) \neq 0$ for $s = \sigma + it$, $\sigma > 1$, $t \in \mathbb{R}$. The same holds for $s = 0$. Hence, the *nontrivial zeroes* of ζ may only lay in the *critical* strip $\mathcal{S} := \{s \in \mathbb{C} : 0 < \sigma < 1\}$. Riemann found three of these zeroes, and the Riemann Hypothesis states that the only nontrivial zeroes in $\mathcal{S}$ are locate on the *critical line* $\mathcal{L} := \{s \in \mathbb{C} : \sigma = \frac{1}{2}\}$. More than 1.5 billion zeroes were found with the help of numerical methods by Odlyzko.

It turns out that $\zeta(s)$ has important applications in the analytic number theory. For example, $\zeta(s)$ is used an important tool to prove the limit known as the Prime Number Theorem ($\mathbb{P}$ denotes the set of prime numbers)

$$\lim_{x \to +\infty} \frac{\pi(x) \ln x}{x} = 1 \, , \tag{3}$$

where $\pi(x) := \sum_{p \in \mathbb{P}: p \leq x} 1$ is related to the known von Mangoldt function, which gives a counting function to the prime numbers. The last result implies that $\zeta(s)$ is a good counting factor for the density of prime numbers in $\mathbb{R}$. To show Eq. (3), we need an analytical study of the nontrivial zeroes of ζ. In doing this, it is convenient to use an entire function the zeroes of which coincide with the nontrivial zeroes of $\zeta(s)$. This function given by

$$\xi(s) := \frac{1}{2} s(s-1)\pi^{-\frac{s}{2}} \Gamma\left(\frac{s}{2}\right) \zeta(s) \, . \tag{4}$$

and is an entire function of order one, i.e. for any $\epsilon > 0$, it satisfies

$$|\xi(s)| = \mathcal{O}_\epsilon(\exp(|s|^{1+\epsilon})), \quad s \in \mathbb{C} \, , \tag{5}$$

and also the relations

$$\xi(s) = \xi(1-s) \qquad , \qquad \xi(s) = \overline{\xi(s)} \, ; \quad \text{for all } s \in \mathbb{C} \, . \tag{6}$$

The Riemann function and the Riemann Hypothesis are also relevant in other areas of mathematics. For example, $\zeta(s)$ is central to establish a connection with the spectral theory known as the Hilbert-Pólya conjecture, which states that the real part of the non-trivial zeroes of $\zeta(s)$ are eigenvalues of a certain Hermitian operators.

Classical Mathematical Results

In our work, among others, we review and prove the following important known results regarding $\zeta(s)$. Below, $B(0;1)$ denotes the unitary complex ball centered at the origin, $\partial B(0;1)$ its boundary and $H(B(0;1))$ the set of entire complex functions on it. ρ is a nontrivial zero of $\zeta(s)$.

Teorema 1. *There are infinitely many nontrivial zeroes of $\zeta(s)$.*

Proof: As the non-trivial zeroes of $\zeta(s)$ are the zeroes of the entire function ξ, then there exists a countable number of zeroes, so we just have to prove that a countable infinite number of nontrivial zeroes. Suppose that there are a finite number of nontrivial zeroes $\rho_1, \rho_2, \ldots, \rho_m$ then $\sum_\rho \frac{1}{|\rho|} = \sum_{k=1}^{m} \frac{1}{|\rho_k|} < \infty$, then as ξ is an entire function of order one $\xi(s) = \exp \mathcal{O}(|s|)$. On the other hand, by Stirling's formula, for $x \in \mathbb{R}$, we have $\log \Gamma\left(\frac{x}{2}\right) \sim \frac{x}{2} \log \frac{x}{2}$, also $\lim_{x \to +\infty} \zeta(x) = 1$ then in (4) we have $\xi(x) \sim \frac{1}{2} x(x-1) e^{\frac{x}{2} \log \frac{x}{2\pi}}$, but this contradicts the fact that $\xi(s) = \exp \mathcal{O}(|s|)$. ■

Teorema 2. *The nontrivial zeroes of $\zeta(s)$ are symmetric with respect to the critical line and the real axis.*

Proof: The statement is a consequence of (6). ■

Teorema 3. *The Riemann hypothesis is true if and only if $\frac{\phi'}{\phi} \in H(B(0;1))$, where $\phi(s) := \xi\left(\frac{s}{s-1}\right)$.*

Proof: Since that $s \mapsto \frac{s}{s-1}$, maps the critical line $\mathcal{L}$ in $\partial B(0;1)$, $\{s \in \mathbb{C} : 0 < \sigma < 1\}$ in $B(0;1)$ and the Theorem 2, we have the Riemann hypothesis is true if the zeroes of ϕ are located in $\partial B(0;1)$, then $\oint_{\partial B(0;1)} \frac{\phi'(s)}{\phi(s)} ds = 0$, consequently $\frac{\phi'}{\phi} \in H(B(0;1))$. The converse is obvious. ■

Teorema 4. *If the Riemann hypothesis is true then the series $\sum_\rho \frac{1}{\rho}$, converges conditionally,*

Proof: As we saw in the proof of Theorem 1, $\sum_\rho |1/\rho|$ diverges. Now if we consider the representation of ξ given by the following product (see [1]) $\xi(s) = \xi(0) \prod_\rho \left(1 - \frac{s}{\rho}\right)$ to calculate the Taylor series of $\frac{\phi'}{\phi}$, we have the coefficient of the independent term is $\sum_\rho \frac{1}{\rho} < \infty$. ■

Riemann Hypothesis and Physics

Now, as a simple among the various available examples, we show some connections between the Riemann Zeta function and a statistical physics model. We take here the set $\mathbb{P} = \{p_1 = 2 < p_2 = 3 < p_3 = 5, p_4 = 7, \ldots\}$ of prime numbers to be increasingly ordered. Now consider a bosonic gas in which the energies of each state are given as follows $E_1 = \log p_1, E_2 = \log p_2, \ldots$. Using the formalism of second quantization we assume each state can be labeled by a natural number n, and we denote it by ψ_n. Now, by the fundamental theorem of arithmetics, every natural number may be decomposed into a product of elements of $\mathbb{P}$. Physically what we mean is that if $n = p_1^{\alpha_1} \cdots p_r^{\alpha_r}$, then in ψ_n we have α_j particles with energy E_j, where $j = 1, 2, \ldots, r$; then the total energy of the state ψ_n is $\mathcal{E}_n = \alpha_1 \log p_1 + \cdots + \alpha_r \log p_r = \log n$, for some real constants α_j. Consequently, the great partition function of a gas of particles in these states is given by (k is the Boltzmann constant, T is the temperature) $\mathcal{Q}_B(T) = \sum_{n=1}^{+\infty} \exp\left(-\frac{\mathcal{E}_n}{kT}\right) = \zeta\left(\frac{1}{kT}\right)$. From this result, we see that the zeroes of ζ can be interpreted as points of phase transitions of this *Primon Gas*, and apply standard physics techniques and results to try to show the Riemann hypothesis. If, otherwise, we consider a fermionic gas, using the Pauli exclusion principle, the great partition function becomes $\mathcal{Q}_F(T) = \sum_{n=1}^{+\infty} |\mu(n)| n^{-1/kT}$ (μ is the Möbius function), which can be of help in this task, as well.

References

[1] EDWARDS, H.M.: *Riemann's Zeta Function,* Harold M. Edwards, 1974.

[2] JULIA, B.: *Statistical theory of numbers*, in Luck, J. M.; Mousa, P.; Waldschmidt, M. (Eds.), Number Theory and Physics, Springer, Berlin, 1990, p. 276.