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FRACTIONAL DIFFERENTIABILITY IN REPRODUCING KERNEL HILBERT SPACES ON THE SPHERE

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We analyze the smoothness of functions in the reproducing kernel Hilbert space (RKHS) generated by a Mercer-like kernel on the sphere, when the generating kernel is sufficiently smooth. The differential here is the fact that smoothness is defined through fractional differentiability, a quite general concept that includes, for instance, the strong Laplace-Beltrami differentiability.

1 Introduction

Let S^m be the unit sphere in $\mathbb{R}^{m+1}$ endowed with its usual normalized surface measure σ_m and consider the usual space $L^p(S^m, \sigma_m)$, for $p \geq 2$. When $p = 2$, the space becomes a Hilbert space with inner product given by

$$\langle f, g\rangle_2 = \int_{S^m} f(x)\overline{g(x)}d\sigma_m(x), \quad f, g \in L^2(S^m, \sigma_m).$$

We will deal with Mercer-like kernels on S^m, i.e., kernels $K : S^m \times S^m \to \mathbb{C}$ having an expansion in the form

$$K(x, y) = \sum_{n\in\mathbb{Z}_+} \lambda_n \phi_n(x)\overline{\phi_n(y)}, \quad x, y \in S^m, \tag{1.1}$$

in which the set $\{\phi_n\}_{n\in\mathbb{Z}_+}$ is an orthonormal sequence in $L^2(S^m, \sigma_m)$, $\{\lambda_n\}_{n\in\mathbb{Z}_+}$ is a sequence of positive real numbers decreasing to 0 and $\sum_{n=1}^{\infty} \lambda_n < \infty$.

If we write $(\mathcal{H}_K, \langle\cdot,\cdot\rangle_K)$ to denote the unique RKHS associated to K, Mercer's theory implies that the set $\{\lambda_n^{1/2}\phi_n\}_{n\in\mathbb{Z}_+}$ is an orthonormal basis for $(\mathcal{H}_K, \langle\cdot,\cdot\rangle_K)$. The basic information mentioned above is to be found in [1] while additional information on positive definiteness, reproducing kernel Hilbert spaces and series representation for kernels can be found in [1, 5, 6].

There are several differentiability concepts for functions on S^m. In this note, the focus will be the fractional derivative, which can be defined via the Laplace-Beltrami series. We recall that the *Fourier-Laplace series* of a function f in $L^2(S^m)$ is $f \sim \sum_{k=0}^{\infty} \mathcal{Y}_k(f)$ in which $\mathcal{Y}_k$, $k = 0, 1, \ldots$, is the orthogonal projection of $L^2(S^m)$ onto the space $\mathcal{H}^k$ of all spherical harmonics of order k in $m+1$ variables. Since $L^2(S^m) = \oplus_{k=0}^{\infty}\mathcal{H}^k$, the series above is , in fact, convergent in $L^2(S^m)$. More details on these facts can be found in [4].

Let r be a positive real and f an element of $L^p(S^m)$. A function g in $L^p(S^m)$ is called the *fractional derivative of order* r of f whenever its Fourier-Laplace series has the form

$$g \sim \sum_{k=1}^{\infty} m^{-r}(k(k+m-1))^r \mathcal{Y}_k(f). \tag{1.1}$$

In that case, we write $g := D^r(f)$. The space of differentiable functions in the sense just defined is then $W_p^r := \{f \in L^p(S^m) : D^r f \in L^p(S^m)\}$, a Banach space when endowed with the norm $\|f\|_{W_p^r} = \|f\|_p + \|D^r f\|_p$. If K is a complex function with domain $S^m \times S^m$ we will write $D^{r,s}K := D_y^s D_x^r K$, in which $D_x^r K$ (respect., $D_y^r K$) means the action of D^r on K, keeping y (respect. x) fixed. Similarly, if $r, s \geq 0$, we define the spaces of differentiable kernels in the sense just defined: $W_p^{r,s} = \{K \in L^p(S^m \times S^m) : D^{r,s}K \in L^p(S^m \times S^m)\}$.

2 Results

A Mercer-like kernel K as in the previous section generates a compact and selfadjoint integral operator on $L^2(S^m, \sigma_m)$, the function ϕ_n in (1.1) being its eigenfunction associated to the eigenvalue λ_n. This information is crucial in the proof of the result below on smoothness of the elements in the basis $\{\lambda_n^{1/2}\phi_n\}_{n\in\mathbb{Z}_+}$ of $\mathcal{H}_K$.

Teorema 2.1. *Let K be a Mercer-like kernel representable as in (1.1). If K belongs to $W_p^{r,0}$, then $\phi_n \in W_p^r$, $n = 0, 1, 2, \ldots$. In addition, the equality*

$$D^r\phi_n(x) = \frac{1}{\lambda_n}\int_{S^m} D^{r,0}K(\cdot, y)\phi_n(y)d\sigma_m(y),$$

holds in S^m a.e.

Under the very same assumptions in Theorem 2.1, if we assume that $K \in W_p^{r,s}$, then

$$D^{r,0}K(x, y) = \sum_{n\in\mathbb{Z}_+} \lambda_n D^r\phi_n(x)\overline{\phi_n(y)}, \quad x, y \in S^m a.e.$$

Teorema 2.2. *Let K be a Mercer-like kernel representable as in (1.1). If K belongs to $W_p^{r,r}$, then $(D^{r,0}K)^x \in \mathcal{H}_K$.*

Replacing the integral representation in the formula of Theorem 2.1 with the inner product of $(\mathcal{H}_K, \langle\cdot,\cdot\rangle_K)$ leads to the following refinement of the previous formula along with some pertinent consequences.

Teorema 2.3. *Let K be a Mercer-like kernel representable as in (1.1). If K belongs to $W_p^{r,r}$, then*

$$D^r\phi_n(x) = \langle\phi_n, (D^{0,r}K)^x\rangle_K, \quad x \in S^m, a.e.$$

Teorema 2.4. *Let K be a Mercer-like kernel representable as in (1.1). If K belongs to $W_p^{r,r}$ then $\mathcal{H}_K$ can be embedded in W_p^r. In addition,*

$$D^r f(x) = \langle f, (D^{0,r}K)^x\rangle_K, \quad f \in \mathcal{H}_K, \quad x \in S^m a.e.$$

Teorema 2.5. *Let K be a Mercer-like kernel representable as in (1.1). If $D^{r,r}K$ exists and the functions $x \to K(x, x)$ and $x \to D^{r,r}K(x, x)$ are in $L^{\frac{p}{2}}(S^m)$, then the embedding $i : \mathcal{H}_K \hookrightarrow C^k(S^m)$ is compact and bounded.*

The references [2,7] consider similar problems but the setting there does not include the spherical one. Reference [2] investigates the case in which the Mercel-like kernel is defined on an open and bounded subset of $\mathbb{R}^{m+1}$, smoothness being defined via usual derivatives.

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